THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010D Advanced Calculus 2019-2020

Solution to Problem Set 7

1. Let $f(x,y) = (f_1(x,y), f_2(x,y)) = (\sqrt{xy}, \sqrt{\frac{y}{x}})$.

   (a) Find the Jacobi matrix $J_f(x,y)$ and evaluate it at the point $(x,y) = (2,8)$.

   (b) By using the linearization of the function f at the point $(x,y) = (2,8)$, approximate $f(1.9, 8.2)$.

   Ans:

   (a) $J_f(x,y) = \frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ -\frac{\sqrt{y}}{2x\sqrt{x}} & \frac{1}{2\sqrt{xy}} \end{bmatrix}$ and so $J_f(2,8) = \begin{bmatrix} 1 & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{8} \end{bmatrix}$.

   (b) We have

$$f(2,8) + J_f(2,8)\begin{bmatrix} x-2 \\ y-8 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{8} \end{bmatrix}\begin{bmatrix} x-2 \\ y-8 \end{bmatrix} = \begin{bmatrix} x + \frac{1}{4}y \\ -\frac{1}{2}x + \frac{1}{8}y + 2 \end{bmatrix}$$

   and so the linearization of $f(x,y)$ is

$$L(x,y) = (x + \frac{1}{4}y, -\frac{1}{2}x + \frac{1}{8}y + 2).$$

   Therefore, $f(1.9, 8.2)$ can be approximated by $L(1.9, 8.2) = (3.95, 2.075)$.

2. Express $\frac{dw}{dt}$ as a function of t if

   (a) $w = x^2 + 2xy$, $x = \cos 2t$, $y = \sin 3t$;

   (b) $w = \ln(xy + yz + zx)$, $x = t^2$, $y = e^t$, $z = \cos t$;

   Ans:

   (a)

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\ &= (2x + 2y)(-2\sin 2t) + (2x)(3\cos 3t) \\ &= -4(x+y)\sin 2t + 6x\cos 3t \\ &= -4(\cos 2t + \sin 3t)\sin 2t + 6\cos 2t \cos 3t \end{aligned}$$

   In matrix notation, we have

$$\begin{bmatrix} \frac{dw}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}.$$

   (b)

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= \left(\frac{y+z}{xy+yz+zx}\right)(2t) + \left(\frac{x+z}{xy+yz+zx}\right)(e^t) + \left(\frac{x+y}{xy+yz+zx}\right)(-\sin t) \\ &= \frac{(t^2 + 2t + \cos t - \sin t)e^t + 2t\cos t - t^2 \sin t}{t^2 e^t + e^t \cos t + t^2 \cos t} \end{aligned}$$

In matrix notation, we have

$$\begin{bmatrix} \dfrac{dw}{dt} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} \dfrac{dx}{dt} \\ \\ \dfrac{dy}{dt} \\ \\ \dfrac{dz}{dt} \end{bmatrix}.$$

3. Express $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ as functions of u and v if

(a) $z = 3e^{2x} \ln y$, $x = \ln(u+v)$, $y = uv$;

(b) $z = xe^y + ye^x$, $x = u + v$, $y = \ln x$.

Ans:

(a)

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= 6e^{2x} \ln y \cdot \frac{1}{u+v} + \frac{3e^{2x}}{y} \cdot v \\ &= 6e^{2\ln(u+v)} \ln(uv) \cdot \frac{1}{u+v} + \frac{3e^{2\ln(u+v)}}{uv} \cdot v \\ &= 6(u+v)\ln(uv) + \frac{3(u+v)^2}{u}. \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= 6e^{2x} \ln y \cdot \frac{1}{u+v} + \frac{3e^{2x}}{y} \cdot u \\ &= 6e^{2\ln(u+v)} \ln(uv) \cdot \frac{1}{u+v} + \frac{3e^{2\ln(u+v)}}{uv} \cdot u \\ &= 6(u+v)\ln(uv) + \frac{3(u+v)^2}{v}. \end{aligned}$$

In matrix notation, we have

$$\begin{bmatrix} \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}.$$

(b)

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\left(\frac{dy}{dx}\frac{\partial x}{\partial u}\right) \\ &= (e^y + ye^x)(1) + (xe^y + e^x)\left(\frac{1}{x}(1)\right) \\ &= 2e^y + e^x\left(y + \frac{1}{x}\right) \\ &= 2(u+v) + e^{u+v}\left(\ln(u+v) + \frac{1}{u+v}\right) \end{aligned}$$

$$
\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\left(\frac{dy}{dx}\frac{\partial x}{\partial v}\right) \\
&= (e^y + ye^x)(1) + (xe^y + e^x)\left(\frac{1}{x}(1)\right) \\
&= 2e^y + e^x\left(y + \frac{1}{x}\right) \\
&= 2(u+v) + e^{u+v}\left(\ln(u+v) + \frac{1}{u+v}\right)
\end{aligned}
$$

In matrix notation, we have

$$
\begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{dy}{dx} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{bmatrix}.
$$

4. If $f(u, v, w)$ is differentiable and $u = x - y$, $v = y - z$ and $w = z - x$, show that

$$
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.
$$

Ans:

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \\
&= \frac{\partial f}{\partial u}\cdot 1 + \frac{\partial f}{\partial v}\cdot 0 + \frac{\partial f}{\partial w}\cdot(-1) \\
&= \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \\
\frac{\partial f}{\partial z} &= -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}
\end{aligned}
$$

Therefore, $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$. In matrix notation, we have

$$
\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.
$$

5. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in \mathbb{R}^3$ and let $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

   Show that for any positive integer n,

$$
\nabla(r^n) = nr^{n-2}\mathbf{r}.
$$

Ans:

Let n be a positive integer, we have $r^n = (x^2+y^2+z^2)^{n/2}$. Then,

$$\begin{aligned}
\nabla(r^n) &= \left(\frac{\partial}{\partial x}r^n, \frac{\partial}{\partial y}r^n, \frac{\partial}{\partial z}r^n\right)\\
&= \left(nx(x^2+y^2+z^2)^{n/2-1}, ny(x^2+y^2+z^2)^{n/2-1}, nz(x^2+y^2+z^2)^{n/2-1}\right)\\
&= \left(nxr^{n-2}, nyr^{n-2}, nzr^{n-2}\right)\\
&= nr^{n-2}(x,y,z)\\
&= nr^{n-2}\mathbf{r}
\end{aligned}$$

6. A function $f(x,y)$ is said to be a **harmonic** if it satisfies the **Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

For $(x,y) \neq (0,0)$, f can be regarded as a function of r and θ with $r > 0$ and $0 \leq \theta < 2\pi$ by

$$f(r,\theta) = f(x(r,\theta), y(r,\theta)),$$

where $x(r,\theta) = r\cos\theta$, $y(r,\theta) = r\sin\theta$ and (r,θ) is called the polar coordinates.

Show that the Laplace equation in polar coordinates can be expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0.$$

Ans: We have

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\\
&= \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}\\
r\frac{\partial f}{\partial r} &= r\cos\theta\frac{\partial f}{\partial x} + r\sin\theta\frac{\partial f}{\partial y}\\
\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) &= \cos\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right) + \sin\theta\frac{\partial f}{\partial y} + r\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)\\
&= \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y} + r\cos\theta\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial r}\right) +\\
&\quad r\sin\theta\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\right)\\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) &= \frac{1}{r}\cos\theta\frac{\partial f}{\partial x} + \frac{1}{r}\sin\theta\frac{\partial f}{\partial y} + \cos\theta\left(\cos\theta\frac{\partial^2 f}{\partial x^2} + \sin\theta\frac{\partial^2 f}{\partial y\partial x}\right) +\\
&\quad \sin\theta\left(\cos\theta\frac{\partial^2 f}{\partial x\partial y} + \sin\theta\frac{\partial^2 f}{\partial y^2}\right)
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\
&= -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y} \\
\frac{\partial^2 f}{\partial \theta^2} &= -r\cos\theta\frac{\partial f}{\partial x} - r\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}\right) - r\sin\theta\frac{\partial f}{\partial y} + r\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial y}\right) \\
&= -r\cos\theta\frac{\partial f}{\partial x} - r\sin\theta\frac{\partial f}{\partial y} - r\sin\theta\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial \theta}\right) + \\
&\qquad r\cos\theta\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta}\right) \\
\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} &= -\frac{1}{r}\cos\theta\frac{\partial f}{\partial x} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial y} - \sin\theta\left(-\sin\theta\frac{\partial^2 f}{\partial x^2} + \cos\theta\frac{\partial^2 f}{\partial y\partial x}\right) + \\
&\qquad \cos\theta\left(-\sin\theta\frac{\partial^2 f}{\partial x\partial y} + \cos\theta\frac{\partial^2 f}{\partial y^2}\right)
\end{aligned}
$$

By the above, $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ is the Laplace equation in polar coordinates.